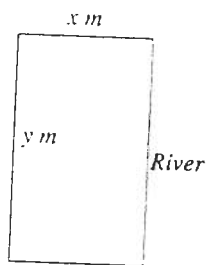
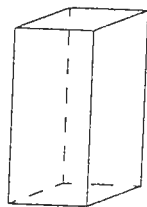


EXERCISE 18.4

1. (a) Find the greatest value of the function $2 + x - 3x^2$.
 (b) Find the greatest value of the function xe^{-x} .
2. (a) Find the least value of the function $8 - 3x + 2x^2$.
 (b) Find the least value of the function $x - \log_e x$.
 (c) Find the least value of the function $4 - \frac{2}{1+x^2}$.
3. Find the range of the function $x \mapsto \frac{x}{1+x^2}$.
4. Two real numbers x and y are such that $x + y = 21$. Find the value of x for which
 (a) the product, xy , is a maximum.
 (b) the product xy^3 is a maximum.
5. If $x + y = 12$, find the minimum value that $x^2 + y^2$ can have.
6. A farmer wishes to fence off a rectangular paddock using an existing stretch of a river as one side. The total length of wiring available is 100 m. Let x m and y m denote the length and width of this rectangular paddock respectively, and let A m² denote its area.



7. A closed rectangular box with square ends is to be constructed in such a way that its total surface area is 400 cm². Let x cm be the side length of the ends and y cm its height.
- (a) Obtain an expression for y in terms of x , stating any restrictions on x .
 - (b) Find the largest possible volume of all such boxes.



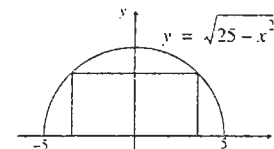
8. A barrel is being filled with water in such a way that the volume of water, V ml, in the barrel after time t seconds is given by

$$V(t) = \frac{2}{3} \left(20t^2 - \frac{1}{6}t^3 \right), \quad 0 \leq t \leq 120$$

- (a) Find the rate of flow into the barrel after 20 seconds.
 - (b) When will the rate of flow be greatest?
 - (c) Sketch the graph of $V(t)$, $0 \leq t \leq 120$.
9. The total cost, SC , for the production of x items of a particular product is given by the linear relation $C = 600 + 20x$, $0 \leq x \leq 100$, whilst its total revenue, SR , is given by $R = x(100 - x)$, $0 \leq x \leq 100$.
 - (a) Sketch the graph of the cost function and revenue function on the same set of axes.
 - (b) Determine the break-even points on your graph.
 - (c) For what values of x will the company be making a positive profit?
 - (d) Find an expression that gives the profit made in producing x items of the product, and hence determine the maximum profit.

10. Find the points on the graph of $y = 9 - x^2$ that are closest to the point $(0, 3)$.

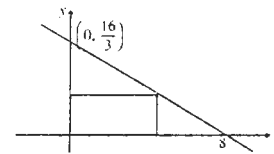
11. A rectangle is bounded by the semi-circle with equation $y = \sqrt{25 - x^2}$, $-5 \leq x \leq 5$ and the x -axis. Find the dimensions of the rectangle having the largest area.



12. A rectangle is bounded by the positive x -axis, the positive y -axis and the line with equation

$$y = \frac{2}{3}(8 - x).$$

Find the dimensions of the rectangle having the largest area.



13. A certificate is to be printed on a page having an area of 340 cm². The margins at the top and bottom of the page are to be 2 cm and, on the sides, 1 cm.

- (a) If the width of the page is x cm, show that the area, A cm² where printed material is to appear is given by

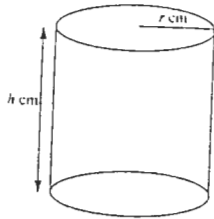
$$A = 348 - \frac{680}{x} - 4x.$$

- (b) Hence, determine the maximum area of print.

14. Find the minimum value of the sum of a positive integer and its reciprocal.

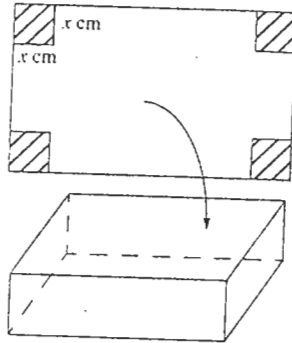
15. A cylinder is to have a surface area of $20\pi \text{ cm}^2$. Determine the dimensions of the cylinder which will have the largest volume.

16. A right circular cylinder of radius $r \text{ cm}$ and height $h \text{ cm}$ is to have a fixed volume of 30 cm^3 .



- (a) Show that the surface area, $A \text{ cm}^2$ of such a cylinder is given by $A = 2\pi r\left(r + \frac{30}{\pi r^2}\right)$.
- (b) Determine the value of r that will yield the minimum surface area.

17. A rectangular container is made by cutting out squares from the corners of a 25cm by 40cm rectangular sheet of metal and folding the remaining sheet to form the container.



- (a) If the squares that are cut out are $x \text{ cm}$ length, show that the volume, $V \text{ cm}^3$ of the container is given by $V = x(25 - 2x)(40 - 2x)$, $0 \leq x \leq \frac{25}{2}$
- (b) What size squares must be cut out in order to maximize the volume of the container?

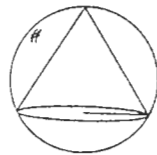
18. A right-circular cone of radius $r \text{ cm}$ contains a sphere of radius 12 cm .

- (a) If the height of the cone is $h \text{ cm}$, express h in terms of r .
- (b) If $V \text{ cm}^3$ denotes the volume of the cone, find an expression for V in terms of r .
- (c) Find the dimensions of the cone with the smallest volume.

19. For a closed cylinder of radius $r \text{ cm}$ and height $h \text{ cm}$, find the ratio $r:h$ which will produce the smallest surface area for a fixed volume.

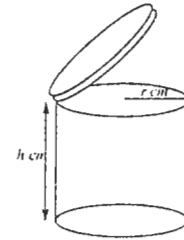
20. For an open cylinder of radius $r \text{ cm}$ and height $h \text{ cm}$, find the ratio $r:h$ which will produce the smallest surface area for a fixed volume.

21. Find the height of a right-circular cone which can be inscribed in a sphere of radius 1 m , if this cone is to have the largest possible volume.



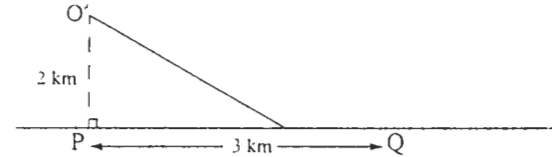
22. A piece of wire 30 cm long is cut into 2 pieces. One of the pieces is bent into a square whilst the other is bent into a circle. Find the ratio of the side length of the square to the radius of the circle which provides the smallest area sum.

23. A cylindrical biscuit tin having a lid 1 cm deep is to have a capacity of $144\pi \text{ cm}^3$.



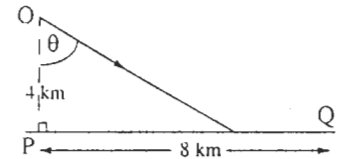
If the cylinder is of radius $r \text{ cm}$ and height $h \text{ cm}$, find the ratio $r:h$ which will give the smallest surface area.

24. The last leg of a triathlon requires that you get from a point O , 2 km from the nearest point P on a straight beach to a point Q , 3 km down the coast.



You may swim to any point on the beach and then run the rest of the way to point Q . If you can swim at a rate of 2 km/hr and run at 5 km/hr , where should you land on the beach in order to reach point Q in the least possible time?

25. A person in a boat 4 km from the nearest point P on a straight beach, wishes to get to a point Q 8 km along the beach from P . The person rows in a straight line to some point on the shore at a constant rate of 5 km/h . Once on the beach the person walks towards Q at a steady rate of 6 km/hr .



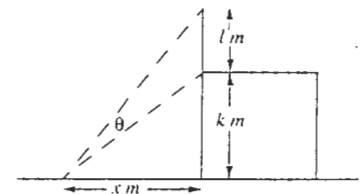
- (a) Show that the total time taken for the trip is

$$T(\theta) = \frac{4 \sec \theta}{5} + \frac{8 - 4 \tan \theta}{6}, \theta \in \left[0, \frac{\pi}{2}\right) \text{ and } \tan \theta \leq 2$$

- (b) Where should the person land so that the trip takes the least amount of time?

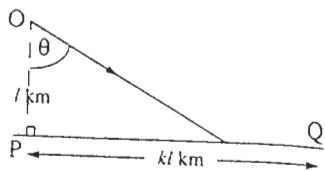
26. A mast l metres tall erected on a building k metres tall, subtends an angle θ at a point on the ground x metres from the base.

- (a) Find an expression for $\tan \theta$ in terms of x .
- (b) Find the value of x that maximizes the value of θ .



[Hint: As $\theta \in \left[0, \frac{\pi}{2}\right)$, θ is a maximum when $\tan \theta$ is a maximum.]

27. A person in a boat l km from the nearest point P on a straight beach, wishes to get to a point Q kl km along the beach from P. The person rows in a straight line to some point on the shore at a constant rate of v km/h. Once on the beach the person walks towards Q at a steady rate of u km/hr where ($v < u$).

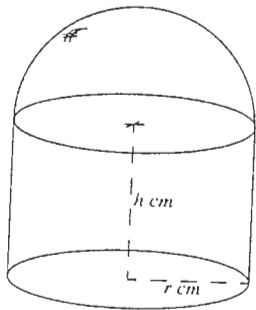


- (a) Show that the total time taken for the trip is

$$T(\theta) = \frac{l \sec \theta}{v} + \frac{kl - l \tan \theta}{u}, \theta \in \left[0, \frac{\pi}{2}\right] \text{ and } \tan \theta \leq k$$

- (b) Show that if the person is to reach point Q in the least possible time, then $\sin \theta = \frac{v}{u}$ where $k \geq c$, c being a particular constant.
- (c) What would happen if $k < c$?

28. A closed tin is to be constructed as shown in the diagram below. It is made up of a cylinder of height h cm and radius base r cm which is surmounted by a hemispherical cap.



- (a) Find an expression in terms of r and h for
- its volume, $V \text{ cm}^3$
 - its surface area, $A \text{ cm}^2$.

- (b) Given that $V = \pi k^3$, $k > 0$, show that its surface area is given by

$$A = 2\pi k^3 \frac{1}{r} + \frac{5\pi}{3} r^2.$$

- (c) Find $r:h$ for A to be a minimum.

29. A ladder is to be carried horizontally around a corner from a corridor a m wide into a corridor b m wide. What is the maximum length that the ladder can be?

